University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 5
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Reading:
105 Notes 3.4-3.7
Hand & Finch 3.4-3.9

1.
A particle of mass m and electric charge q is situated in an alternating electric field directed along the x axis: $E_x = E_0 \cos\omega t$. The particle also experiences a force in the x direction proportional to the *third derivative* of its x coordinate:

$$F_\alpha = +\alpha \frac{d^3x}{dt^3} \ ,$$

where α is a positive constant. [This model gives an approximate description of a charged particle that scatters radiation.]

Find the amplitude and phase of the particle's oscillation in the steady state.
Solution:
The particle feels two forces: $F_{\text{elec}} = qE = qE_0 \cos\omega t$, and $F_\alpha = \alpha\dddot{x}$. So its equation of motion is

$$m\ddot{x} = \alpha\dddot{x} + qE_0\, e^{i\omega t}$$

(We can replace $\cos\omega t$ by $e^{i\omega t}$ because their real parts are the same.) As usual, we solve this kind of equation by guessing that the answer is of the form $x = \mathrm{Re}\left(Ae^{i\omega t}\right)$. Then the equation of motion becomes

$$-mA\omega^2 = qE_0 - i\alpha A\omega^3$$

Solve this equation for A:

$$A = \frac{qE_0}{-m\omega^2 + i\alpha\omega^3} = -\frac{qE_0}{\omega^2}\frac{m + i\alpha\omega}{m^2 + \alpha^2\omega^2}$$

So A is a complex number of the form $A = |A|\, e^{i\varphi}$, with amplitude and phase

$$|A| = \frac{qE_0}{\omega^2\sqrt{m^2 + \alpha^2\omega^2}} \ , \quad \varphi = \pi + \arctan\Bigl(\frac{\alpha\omega}{m}\Bigr) \ .$$

2.
Consider an extremely underdamped oscillator $(\omega_0/\gamma \equiv Q \gg 1)$.
(**a**)
Suppose that the oscillator is undriven, but initially it is excited. How many oscillation periods are required for the energy stored in the oscillator to diminish by a factor of e?
Solution:
The general solution to the undriven underdamped oscillator is

$$\begin{aligned} x(t) &= Be^{-\gamma t/2}\cos\left(\omega_\gamma t + \beta\right) \ , \text{ where} \\ \omega_\gamma^2 &\equiv \omega_0^2 - \frac{\gamma^2}{4} \\ &= \omega_0^2\Bigl(1 - \frac{1}{4Q^2}\Bigr) \\ &\to \omega_0^2 \ \text{ for } Q \gg 1 \ . \end{aligned}$$

The total energy in the oscillator is the sum of kinetic and potential energy terms:

$$\begin{aligned} E &= \tfrac{1}{2}kx^2 + \tfrac{1}{2}m\dot{x}^2 \\ &= \tfrac{1}{2}m\omega_0^2 x_{\max}^2 \ , \end{aligned}$$

where as usual $\omega_0^2 \equiv k/m$, and we have used the fact that $\dot{x} = 0$ when x is at its maximum displacement $x_{\max}$. From the general solution, $x_{\max} \propto \exp\left(-\gamma t/2\right)$, so $E \propto \exp\left(-\gamma t\right)$. Therefore the time τ required for E to diminish by a factor e is $\tau = 1/\gamma$. In this time, the number N of periods is

$$N = \frac{\tau}{2\pi/\omega_\gamma} \approx \frac{\gamma^{-1}}{2\pi/\omega_0} = \frac{Q}{2\pi} \ .$$

(**b**)
Instead suppose that the oscillator is driven at

resonance. What is the ratio of the energy stored in the oscillator to the work done by the driving force in one oscillation period?

Solution:

As usual substitute $x \equiv \mathrm{Re}\big(Ae^{i\omega t}\big)$, and choose to solve the complex equation. When the driving force is $F_0 \cos \omega t$, this yields

$$-\omega^2 A + i\gamma\omega A + \omega_0^2 A = F_0/m \ .$$

At resonance ($\omega \equiv \omega_0$),

$$A = \frac{-iF_0}{\gamma\omega_0 m} \ .$$

Using the arguments in part (**a**), the energy in the oscillator is

$$\begin{aligned} E &= \tfrac{1}{2} m\omega_0^2 x_{\max}^2 \\ &= \tfrac{1}{2} m\omega_0^2 \frac{F_0^2}{\gamma^2\omega_0^2 m^2} \\ &= \frac{F_0^2}{2\gamma^2 m} \ . \end{aligned}$$

To determine the energy dissipated in one period, we integrate the work W done by the driving force:

$$\begin{aligned} W &= \oint F_x \, dx \\ &= \oint F_x \, \frac{dx}{dt} dt \\ &= \oint F_x v_x \, dt \ . \end{aligned}$$

Using our determination of A, we know $x(t)$ and therefore $v(t)$:

$$\begin{aligned} x(t) &= \mathrm{Re}\Big(\frac{-iF_0}{\gamma\omega_0 m} e^{i\omega_0 t}\Big) \\ &= \frac{F_0}{\gamma\omega_0 m} \sin\omega_0 t \\ v_x(t) &= \frac{F_0}{\gamma m} \cos\omega_0 t \ . \end{aligned}$$

Plugging $v_x(t)$ into the integral, and recalling that the square of any circular function has an average value of $\frac{1}{2}$ over one period T,

$$\begin{aligned} W = \oint F_x v_x \, dt &= \oint dt \, F_0 \cos\omega_0 t \, \frac{F_0}{\gamma m} \cos\omega_0 t \\ &= \frac{T}{2} \frac{F_0^2}{\gamma m} \\ &= \frac{2\pi}{2\omega_0} \frac{F_0^2}{\gamma m} \\ &= \frac{\pi}{Q} \frac{F_0^2}{\gamma^2 m} \ . \end{aligned}$$

Comparing W to E,

$$\frac{E}{W} = \frac{Q}{2\pi} \ ,$$

the same ratio obtained in part (**a**).

3.

Consider an undriven oscillator satisfying the initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$. Find $x(t)$ when the oscillator is...

(**a**)

...slightly underdamped ($\omega_0/\gamma \equiv Q = \frac{1}{\sqrt{2}}$).

Solution:

The general solution to the undriven underdamped oscillator is

$$\begin{aligned} x(t) &= Be^{-\gamma t/2} \cos\left(\omega_\gamma t + \beta\right) \ , \text{ where} \\ \omega_\gamma^2 &\equiv \omega_0^2 - \frac{\gamma^2}{4} \\ &= \omega_0^2 \Big(1 - \frac{1}{4Q^2}\Big) \\ &= \frac{\omega_0^2}{2} \ \text{ when } \ Q = \frac{1}{\sqrt{2}} \\ x(t) &= Be^{-\omega_0 t/\sqrt{2}} \cos\big(\frac{\omega_0 t}{\sqrt{2}} + \beta\big) \ . \end{aligned}$$

Applying the boundary condition that the initial velocity vanishes,

$$\begin{aligned} 0 = \dot{x}(0) &= -\frac{\omega_0}{\sqrt{2}} B\cos\beta - B\frac{\omega_0}{\sqrt{2}} \sin\beta \\ \Rightarrow \cos\beta &= -\sin\beta \\ \beta &= -\frac{\pi}{4} \ . \end{aligned}$$

Finally, setting the initial displacement equal to x_0,

$$\begin{aligned} x_0 = x(0) &= B\cos\beta = B\cos\big(-\frac{\pi}{4}\big) \\ B &= \sqrt{2} x_0 \ . \end{aligned}$$

Putting it all together,

$$x(t) = \sqrt{2}x_0 e^{-\omega_0 t/\sqrt{2}} \cos\big(\frac{\omega_0 t}{\sqrt{2}} - \frac{\pi}{4}\big) \ .$$

(**b**)
...slightly overdamped ($\omega_0/\gamma \equiv Q = \frac{6}{13}$).
Solution:
The general solution to the undriven overdamped oscillator is

$$\begin{aligned} x(t) &= C_+ e^{-\gamma_+ t} + C_- e^{-\gamma_- t}\ , \text{ where} \\ \gamma_\pm &\equiv \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \\ Q &= \tfrac{6}{13} \\ \Rightarrow \gamma &= \tfrac{13}{6}\omega_0 \\ \gamma_\pm &= \omega_0\big(\tfrac{13}{12} \pm \sqrt{(\tfrac{13}{12})^2 - 1}\big) \\ &= \omega_0\big(\tfrac{13}{12} \pm \tfrac{5}{12}\big) \\ \gamma_+ &= \tfrac{3}{2}\omega_0 \\ \gamma_- &= \tfrac{2}{3}\omega_0\ . \end{aligned}$$

Applying the boundary condition that the initial velocity vanishes,

$$\begin{aligned} 0 = \dot{x}(0) &= -\gamma_+ C_+ - \gamma_- C_- \\ &= -\tfrac{3}{2}\omega_0 C_+ - \tfrac{2}{3}\omega_0 C_- \\ C_+ &= -\tfrac{4}{9} C_-\ . \end{aligned}$$

Finally, setting the initial displacement equal to x_0,

$$\begin{aligned} x_0 = x(0) &= C_+ + C_- \\ &= \big(-\tfrac{4}{9} + 1\big)C_- \\ C_- &= \tfrac{9}{5}x_0 \\ C_+ &= -\tfrac{4}{5}x_0\ . \end{aligned}$$

Putting it all together,

$$x(t) = \frac{x_0}{5}\Big(9\exp\big(-\tfrac{2}{3}\omega_0 t\big) - 4\exp\big(-\tfrac{3}{2}\omega_0 t\big)\Big)\ .$$

4.
Consider a critically damped oscillator ($\omega_0/\gamma \equiv Q = \frac{1}{2}$) that remains at rest at $x = 0$ for $t < 0$, but is driven at resonance by a force F_x such that

$$\frac{F_x}{m} = G \sin \omega_0 t$$

for $t > 0$, where G is a constant. Find $x(t)$. [*Hint:* It is somewhat easier to solve this problem directly (matching boundary conditions at $t = 0$) than to use a Green function.]
Solution:
For time $t > 0$ the equation we're solving is of the form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = G \sin \omega t,$$

which we've solved before. The particular solution is most easily written in complex notation as $x_p(t) = \mathrm{Re}\left(Ae^{i\omega t}\right)$, where

$$A = \frac{-iG}{(\omega_0^2 - \omega^2) + i\gamma\omega}\ .$$

Substituting $\gamma = 2\omega_0$ (critical damping) and $\omega = \omega_0$ (driven at resonance) yields

$$A = -\frac{G}{\gamma\omega_0} = -\frac{G}{2\omega_0^2}\ .$$

We need to add a homogeneous solution of the form $x_h = D_1 e^{-\omega_0 t} + D_2 t e^{-\omega_0 t}$ to this to make it meet the initial conditions. The conditions are

$$\begin{array}{lcr} x(0) = 0, & \text{so} & A + D_1 = 0 \\ \dot{x}(0) = 0, & \text{so} & -\omega_0 D_1 + D_2 = 0 \end{array}$$

Therefore

$$\begin{aligned} D_1 &= \frac{G}{2\omega_0^2} \\ D_2 &= \frac{G}{2\omega_0}\ . \end{aligned}$$

Putting it all together,

$$x(t) = \frac{G}{2\omega_0^2}\Big(e^{-\omega_0 t} + \omega_0 t e^{-\omega_0 t} - \cos\omega_0 t\Big)\ .$$

5.
Woofer design. With compact discs well established as a recording medium, loudspeaker distortion is the last major barrier to true sound reproduction. A woofer in a sealed box ("acoustic suspension") is the simplest type to analyze.

The motion of the cone of mass m is governed by the equation

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\omega t\ ,$$

where F_0 is constant if the amplifier output resistance or the voice coil resistance is excessive (not a typical assumption, but one we will make here for simplicity). The average sound intensity is proportional to the average (acceleration)^2 of the cone. The damping factor b is proportional to the strength of the magnetic "motor" – the magnet and voice coil assembly. The spring constant k is inversely proportional to the volume of air sealed in the box.

(**a**)
Try to think up a simple mechanical test that you can perform in the showroom (when the salesperson is looking the other way) to see whether the cone is underdamped or overdamped.

Solution:
Just do something to displace the cone from its equilibrium position (by pushing on it or something). If it is overdamped, it will return straight to where it started. If it is underdamped, it will oscillate first.

(**b**)
Suppose the assembly goes through resonance at $\nu_0=50$ Hz with $Q=1$. (These are typical specifications for a medium quality classical music speaker.) By what factor will the sound intensity vary at 25 Hz? 100 Hz?

Solution:
Remember that the resonant angular frequency $\omega_0=\sqrt{k/m}$. How is Q defined? If you weren't given both γ, the damping coefficient, and ω_0, you could measure Q from $Q=\omega_0/\Delta\omega$, where $\Delta\omega$ is the full width at half maximum of the function $\omega^2\,|A|^2$ (whose maximum is at ω_0). In our case, we do know $\gamma\equiv b/m$ and ω_0, so it is simpler to use the definition $Q=\omega_0/\gamma$. (If you weren't given γ you could measure it from the solution to the undriven oscillator: $x(t)=e^{-\gamma t/2}\cos\omega t$.) In our case, $Q=1$, so $\gamma=\omega_0$.

Now, the equation of motion for our speaker is

$$m\ddot{x}+b\dot{x}+kx=F_0\cos\omega t\ .$$

The steady-state solution must be of the form $x(t)=\mathrm{Re}(Ae^{i\omega t})$, with

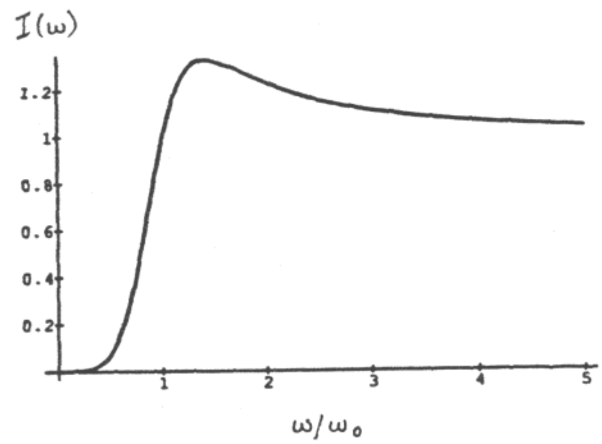
$$\left(-m\omega^2+ib\omega+k\right)A=F_0$$

$$A=\frac{F_0}{(k-m\omega^2)+ib\omega}=\frac{F_0/m}{(\omega^2-\omega_0^2)+i\gamma\omega}$$

The sound intensity is proportional to the average acceleration squared, which is proportional to $\omega^4\,|A|^2$. That is,

$$I(\omega)=\frac{C\omega^4}{\left(\omega^2-\omega_0^2\right)^2+\gamma^2\omega^2}$$
$$=\frac{C\omega^4}{\left(\omega^2-\omega_0^2\right)^2+\omega_0^2\omega^2}\ ,$$

where C is a constant. This function is plotted below for your amusement.



The relevant facts about it for this problem, though, are that

$$\frac{I(\frac{1}{2}\omega_0)}{I(\omega_0)}=\frac{1}{13}=0.0769$$
$$\frac{I(2\omega_0)}{I(\omega_0)}=\frac{16}{13}=1.231\ .$$

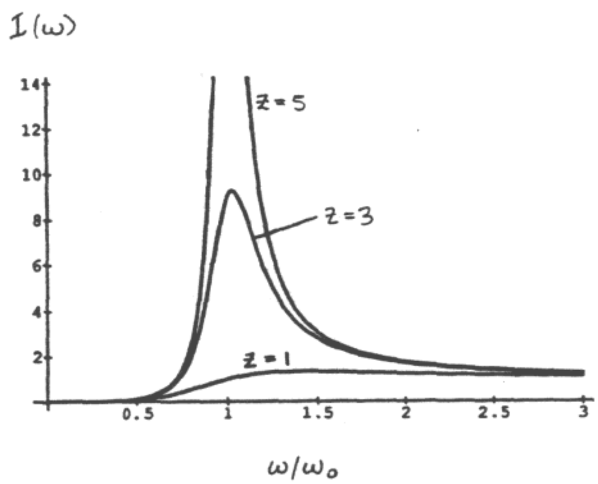
(**c**)
Sketch the effect upon smoothness of bass response of greatly increasing the cone area (to an inexperienced buyer, this often increases the speaker's apparent value). [*Hint*: Make reasonable assumptions concerning the dependence of m and k on the cone area.]

Solution:

What does increasing the cone size do to ω_0 and γ? Well, if $\mathcal{A}$ is the cone area, then $k \propto \mathcal{A}$ (because $k \propto F$, and force is pressure times area), and $m \propto \mathcal{A}$, so $\omega_0 = \sqrt{k/m}$ is independent of $\mathcal{A}$, and $\gamma = b/m \propto \mathcal{A}^{-1}$. So if we let $z = \mathcal{A}/\mathcal{A}_{\text{initial}}$ be the factor by which the area is expanded, then $\gamma = \omega_0/z$, and our expression for the sound intensity becomes

$$
\begin{aligned}
I(\omega) &= \frac{C\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2} \\
&= \frac{C\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \omega_0^2\omega^2/z^2} .
\end{aligned}
$$

Now we can plot $I(\omega)$ for various values of z to see how smooth the response is.



From the graph, we see that the audio response curve gets much less smooth *vs.* frequency as z increases. (The physical explanation for this is that the Q of the oscillator goes up as the cone size increases, so the peak is sharper.) You could come up with all kinds of ways to make this statement more precise. One would be to define the "smoothness" ratio R to be $I(2\omega_0)/I(\omega_0)$ (so that $R = 1$ is the desirable state), and plot R vs. z. By this method of measuring, you'd find that the smoothness indeed does get worse as the cone size increases.

6.
Obtain the Fourier series that represents the function

$$
\begin{aligned}
F(t) &= 0 \;\; (-\frac{2\pi}{\omega} < t < 0) \\
&= F_0 \sin\omega t \;\; (0 < t < \frac{2\pi}{\omega}) .
\end{aligned}
$$

Solution:
Remember: Any function $F(t)$ that repeats itself with period $T = 2\pi/\Omega$ and whose average value is zero can be expanded in a series

$$
\begin{aligned}
F(t) &= \sum_{n=1}^{\infty} \left(f_n \sin n\Omega t + g_n \cos n\Omega t\right) \text{ , where} \\
f_n &= \frac{\Omega}{\pi} \int_0^T F(t) \sin n\Omega t \\
g_n &= \frac{\Omega}{\pi} \int_0^T F(t) \cos n\Omega t .
\end{aligned}
$$

In our case, $F(t)$ repeats itself with period $T = 4\pi/\omega$, so $\Omega = \omega/2$. Then

$$
\begin{aligned}
f_n &= \frac{\omega}{2\pi} \int_{-2\pi/\omega}^{2\pi/\omega} F(t) \sin \tfrac{1}{2} n\omega t \, dt \\
&= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} F_0 \sin\omega t \, \sin \tfrac{1}{2} n\omega t \, dt .
\end{aligned}
$$

The best way to do integrals like these is to look them up, but if you're too proud, you can also calculate them by using the angle addition formulæ (in reverse) to write the integrand as a difference of two cosines. In either case, the answer comes out to be

$$
f_2 = \tfrac{1}{2} F_0 \qquad \text{and} \qquad f_n = 0 \quad \text{for } n \neq 2 .
$$

Now on to the g_n's: The relevant integral this time is

$$
\begin{aligned}
g_n &= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} F_0 \sin\omega t \, \cos \tfrac{1}{2} n\omega t \, dt \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi(4-n^2)} F_0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

So the answer is

$$
F(t) = \tfrac{1}{2} F_0 \sin\omega t + \sum_{n \text{ odd}} \frac{4}{\pi\left(4 - n^2\right)} F_0 \cos \tfrac{1}{2} n\omega t
$$

7.
Consider a damped oscillator (as usual, characterized by γ, ω_0, and m) driven by a periodic force F. During one period,

$$-\frac{2\pi}{\omega} < t < \frac{2\pi}{\omega} \ ,$$

F is taken to be equal to $F(t)$ in problem (**6.**); before and afterward, it simply repeats itself.

Find $x(t)$ for this oscillator. You may assume that any transient effects, due to the driving force having been turned on at $t = -\infty$, have damped out.

Solution:
Expressing $F(t)$ in terms of the Fourier series obtained in the previous problem, we need a particular solution to the differential equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{2m}\sin\omega t + \sum_{n\ \mathrm{odd}} \frac{4}{\pi(4-n^2)}\frac{F_0}{m}\cos\tfrac{n}{2}\omega t \ .$$

A linear oscillator must simultaneously respond at all frequencies at which it is driven, so we seek solutions of the form

$$x(t) = \mathrm{Re}\Big(Ae^{i\omega t} + \sum_{n\ \mathrm{odd}} B_n e^{i\frac{n}{2}\omega t}\Big) \ .$$

We substitute this form of x in the differential equation, and, as usual, we choose to solve the complex version of it. To do so, on the left *vs.* right-hand sides, we equate the coefficients of each factor of the form $\exp(i\Omega t)$, where Ω is equal either to ω or to $\frac{n}{2}\omega$ for any odd n. This yields the set of equations

$$A = \frac{-iF_0/2m}{(\omega_0^2 - \omega^2) + i\gamma\omega} \qquad B_n = \frac{4}{\pi\,(4-n^2)}\,\frac{F_0/m}{\left(\omega_0^2 - (\frac{n}{2})^2\omega^2\right) + i\gamma\frac{n}{2}\omega} \ .$$

(Note that A is of the same form as in problem (**4.**), and that the B_n are of standard form except for their n-dependent coefficients.) The solution is thus

$$x(t) = \mathrm{Re}\Big(\frac{-iF_0/2m}{(\omega_0^2 - \omega^2) + i\gamma\omega}e^{i\omega t} + \sum_{n\ \mathrm{odd}} \frac{4}{\pi\,(4-n^2)}\,\frac{F_0/m}{(\omega_0^2 - (\frac{n}{2})^2\omega^2) + i\gamma\frac{n}{2}\omega}e^{i\frac{n}{2}\omega t}\Big) \ .$$

If you are fond of such things, you can reexpress the contents of the large parentheses as a sum of purely real or imaginary parts, or rewrite $x(t)$ in terms of many different cosines and phases.

8.
Derive the Green function for an *overdamped* oscillator initially at rest at the origin. [*Hint*: Use the method of 105 Notes sections 3.5 and 3.6.]
Solution:
The equation for the Green function (call it X_g) is $\ddot{X}_g + \gamma\dot{X}_g + \omega_0^2 X_g = \delta(t)$. For $t > 0$ the right-hand side is zero, so we can just write down a solution to the homogeneous equation:

$$X_g = A_1\,e^{\alpha_+ t} + A_2\,e^{\alpha_- t} \ \text{ where}$$
$$\alpha_\pm = -\tfrac{1}{2}\gamma \pm \tfrac{1}{2}\sqrt{\gamma^2 - 4\omega_0^2} \ ,$$

and A_1 and A_2 are constants determined by the initial conditions. What are the initial conditions? Well, $X_g(0) = 0$, because the oscillator hasn't had any time to move immediately after the impulse, and $\dot{X}_g(0) = 1$, because the impulse per unit mass supplied by the δ-function is $\Delta v = \int \delta(t)\,dt = 1$. Solve these two equations and you get

$$A_1 = -A_2 = \frac{1}{\alpha_+ - \alpha_-} = \frac{1}{\sqrt{\gamma^2 - 4\omega_0^2}} \ .$$

So the Green function is

$$X_g(t) = \frac{1}{\sqrt{\gamma^2 - 4\omega_0^2}}\left(e^{\alpha_+ t} - e^{\alpha_- t}\right) \ .$$